

Light-like tachyon condensation in Open String Field Theory

Simeon Hellerman and Martin Schnabl

Institute for Advanced Study, Princeton, NJ 08540 USA

E-mail: `simeon at ias.edu`, `schnabl at ias.edu`

Abstract

We use open string field theory to study the dynamics of unstable branes in the bosonic string theory, in the background of a generic linear dilaton. We find a simple exact solution describing a dynamical interpolation between the perturbative vacuum and the recently discovered nonperturbative tachyon vacuum. In our solution, the open string tachyon increases exponentially along a null direction, after which nonlinearities set in and cause the solution to asymptote to a static state. In particular, the wild oscillations of the open string fields which plague the time-like rolling tachyon solution are entirely absent. Our model thus represents the first example proving that the true tachyon vacuum of open string field theory can be realized as the endpoint of a dynamical transition from the perturbative vacuum.

1 Introduction

Much has been learned about tachyon condensation [1, 2, 3] in the past decade but many interesting questions remain. One of the most important problems at the moment is to better understand the *dynamics* of the tachyon condensation process.

In the context of closed string tachyon condensation intriguing dynamical solutions have been constructed recently [4] - [11] (also [12, 13]) using worldsheet sigma model techniques. The key observation of those papers was, that there are completely controlled solutions with finite tachyon condensate depending on a single light-cone coordinate X^+ . Our motivation in the present paper is to construct analogous solutions for the open string.

For purposes of studying closed string dynamics, the sigma-model methods are most popular, since the closed string field theory is somewhat complicated. For open strings the field-theoretic point of view is much better developed. Especially after the recent revival following [14] it is hoped that open string field theory (OSFT) could become the most comprehensive and fine tool for studying tachyon dynamics. Furthermore, we expect that the comparison between CFT and OSFT points of view should yield valuable connections.

Several time-dependent tachyon-condensation solutions in OSFT have been found [15, 16], but they only confirmed some puzzling behavior found earlier via level truncation [26, 28, 30] and in the p -adic string model. Rather than revisiting these solutions and trying to extract new insights from them, we focus on a different class of solutions.

We construct and study solutions where the tachyon depends on a single light-cone coordinate $X^+ = (X^0 + X^1)/\sqrt{2}$. These configurations can be thought as waves sweeping through the D-brane worldvolume leaving behind the true vacuum. After all, this might be a more natural process than homogenous time dependent decay. For somewhat technical reasons we are going to study the whole process in an arbitrary linear dilaton background. As it turns out, the nontrivial dilaton gradient smoothes out the solution and in particular allows the null tachyon to relax asymptotically to the true vacuum, a situation that is impossible for a time-like tachyon trajectory in a background with constant dilaton.

Light-like vs. time-like tachyon solutions – a field theory model

Qualitatively, this contrast can be understood in a simple field theory model for the tachyon and background dilaton. Defining $g = g_o^2 = \exp(\Phi)$, consider the simple field theory action

$$S = \int d^D x e^{-\Phi} \left[-\frac{1}{2}(\partial\mathcal{T})^2 - U(\mathcal{T}) \right] , \quad (1.1)$$

where \mathcal{T} is the tachyon, Φ is the dilaton and $U(\mathcal{T})$ is the tachyon potential. We treat Φ as a nondynamical background field. This is justified as a model for string theory in the weak coupling limit, since the dilaton kinetic term is parametrically larger than that of the tachyon

for $g \rightarrow 0$, suppressing the backreaction of the open string fields on the dilaton by an infinite amount as the coupling is turned off.

First consider the case of constant dilaton and time-like tachyon $\mathcal{T} = \mathcal{T}(x^0)$. The equation of motion for the tachyon is

$$\left(\frac{\partial}{\partial x^0}\right)^2 \mathcal{T} = -U'(\mathcal{T}) . \quad (1.2)$$

If we assume U has an unstable maximum at $\mathcal{T} = 0$ and a stable minimum at $\mathcal{T} = \mathcal{T}_0$, then the generic behavior will be for \mathcal{T} to roll initially away from 0 and towards the true minimum.

However in the time-like case it is impossible for \mathcal{T} to approach the minimum asymptotically at late times: instead, the field will oscillate about its minimum with an amplitude that never decreases. More to the point, conservation of energy alone forbids the tachyon from settling into its minimum as $x^0 \rightarrow +\infty$.

Now consider the second case, where the dilaton is linear as a function of space, $\Phi = V_\mu x^\mu$, and we assume \mathcal{T} depends on a null coordinate $x^+ \equiv \frac{1}{\sqrt{2}}(x^0 + x^1)$, rather than a time-like coordinate. Let us further assume that $V^+ = -V_- > 0$. With this *ansatz*, the equation of motion for \mathcal{T} is

$$\frac{\partial}{\partial x^+} \mathcal{T} = -\frac{1}{V^+} U'(\mathcal{T}) \quad (1.3)$$

The initial behavior of the tachyon is similar to that of the time-like case: at early times, the tachyon rolls away from $\mathcal{T} = 0$ with exponentially increasing amplitude, although here the rate of increase is set by $\frac{1}{V^+}|U''(0)|$, rather than by $|U''(0)|^{\frac{1}{2}}$ as it is in the time-like case. In the light-like case, however, evolution of \mathcal{T} is given not by Hamiltonian dynamics but by *gradient flow* with respect to the potential $\mathcal{U}(\mathcal{T})$, which in some sense is the exact opposite of Hamiltonian behavior. Once \mathcal{T} is in the basin of attraction of the true vacuum at $\mathcal{T} = \mathcal{T}_0$, gradient flow dictates that it will asymptote to \mathcal{T}_0 at late times, with exponentially decreasing distance.

Despite the close connection between the light-like and time-like cases, the difference in behaviors is striking. In the time-like case, energy conservation forbade the field from asymptoting to the true vacuum at late times. In the light-like case (with dilatonic damping), the gradient flow *forces* the tachyon to asymptote to the true vacuum at late times.

These two cases both have close analogs in OSFT. The case of time-like rolling is by now quite well-studied, using boundary state/ σ -model methods [62, 64, 63, 66, 67, 68, 70, 71, 65], effective field theory models [69], and OSFT techniques [15, 16, 26]. The most notable common feature of these approaches is the late time behavior of the solution: in the limit $x^0 \rightarrow +\infty$, the tachyon (and other fields) do *not* reach the classical ground state of the system that was found in [14]. In the boundary state description, the energy density stays strictly constant, and equal to that of the perturbative vacuum, rather than the true vacuum. In the corresponding OSFT solution, the string fields oscillate violently about the true vacuum, but again with a constant nonzero amount of total energy density stored in their motion. As demonstrated in

the field-theoretic toy model above, it is completely inevitable that the open string fields will fail to reach their minimum through a spatially homogeneous solution, as long as the closed string background is spatially homogeneous and static : *Conservation of energy is a universal obstruction to interpolating dynamically between the perturbative vacuum and the true vacuum.*

In the light-like case, which we shall describe in this paper, the obstruction is avoided by considering the OSFT in linear dilaton backgrounds, which are necessarily either time-dependent or spatially inhomogeneous. (For spatially inhomogeneous backgrounds, the energy density need not be constant, and the integrated energy density can diverge, so that there is no meaningful conserved energy.) The tachyon and other fields smoothly approach the true vacuum at late times, due to the dissipative effect of the background dilaton gradient.

Relation to other work

The key feature that allows our solution to be derived in closed form is the *ansatz* that the OSFT configuration should preserve a null translational symmetry when the open string fields are written in string-frame normalization. That is, with the fields normalized so that the kinetic term goes as g_o^{-2} , the open string field components depend only on the null coordinate x^+ and are independent of x^- .

There is a facial similarity between such solutions and a large class of string backgrounds whose solvable classical and quantum dynamics derive from the existence of a covariantly constant null Killing field. Such backgrounds include pure gravitational waves [72], pp-waves with Ramond fluxes [73, 74], null-branes [75] and null orbifolds [76, 77] and more general examples in which moduli vary arbitrarily along null directions [78].

The null symmetry of the models above gives rise to tractable classical and quantum properties, while allowing for time-dependence and dynamical phenomena that could not arise in the static case. The OSFT solutions we consider are somewhat similar to these pp-wave solutions mentioned above in that the open string degrees of freedom respect a null translational symmetry, but with the important difference that the null symmetry is broken by the dilaton.

The breaking is essential: the solutions we study would not exist in a background with constant dilaton and have no analog there, as their dynamics is fundamentally dissipative. However the particularly simple tree-level coupling of strings to the dilaton allows many of the simplifying aspects of the null isometry to survive, particularly when classical solutions are considered. This type of simplification has been exploited in many cases in closed string theory in the σ -model approach [12, 13],[4] -[11] to generate solutions of string theory that are fully α' -exact. The solutions we construct here in the OSFT system have many direct parallels with those.

Outline of the paper

This paper is organized as follows: In section 2 we review useful facts about linear dilaton CFT. In section 3 we discuss the formulation of OSFT in the background of a linear dilaton. In section 4, we construct an exact light-like rolling tachyon solution in OSFT and show that at late times the solution settles into the true vacuum. We also calculate the behavior of the light-like tachyon solution in the conventional level-truncation of the cubic OSFT. Section 5 discusses the null tachyon condensate in terms of the σ -model/boundary state formalism. In section 6 we study the same solution in a p-adic string model that can be viewed as the lowest-level approximation to vacuum string field theory (VSFT). Section 7 concludes with observations on purely time-like solutions that have not been made before.

2 Review of linear dilaton CFT

Linear dilaton CFT on a worldsheet Σ with boundary $\partial\Sigma$ is based upon an action

$$S_{ws} = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{h} \left(\eta_{\mu\nu} h^{ab} \partial_a X^\mu \partial_b X^\nu + \alpha' \mathcal{R} V_\mu X^\mu \right) + \frac{1}{2\pi} \int_{\partial\Sigma} K V_\mu X^\mu, \quad (2.1)$$

where h_{ab} is the intrinsic worldsheet metric, \mathcal{R} its curvature, and K is the extrinsic curvature of the boundary which integrates to 2π along the boundary $\partial\Sigma$. For applications to classical open string field theory we shall restrict to worldsheets of disk topology. Gauge fixing the diffeomorphism and Weyl invariance (assuming the central charge to be zero after the addition of the ghost sector) we can set the metric to the canonical form $ds^2 = dzd\bar{z}$ on the upper half plane and the worldsheet action takes a simple form

$$S_{ws} = \frac{1}{2\pi\alpha'} \int_{\Sigma} \eta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu. \quad (2.2)$$

The energy-momentum tensor coming from (2.1) is given by

$$T_{zz}(z) = -\frac{1}{\alpha'} \eta_{\mu\nu} : \partial X^\mu \partial X^\nu : + V_\mu \partial^2 X^\mu \quad (2.3)$$

and similarly for the antiholomorphic component. Note that the second term can be viewed as an allowed 'improvement' of a canonical energy momentum tensor arising from (2.2).

The action (2.2) is invariant under the infinitesimal conformal transformation $([33])z' = z + \varepsilon v(z)$

$$\delta X^\mu(z, \bar{z}) = -\varepsilon v(z) \partial X^\mu(z, \bar{z}) - \varepsilon v(z)^* \bar{\partial} X^\mu(z, \bar{z}) - \frac{\varepsilon}{2} \alpha' V^\mu (\partial v(z) + (\partial v(z))^*) \quad (2.4)$$

or its finite version $z' = f(z)$

$$X^\mu(z, \bar{z}) \rightarrow f \circ X^\mu(z, \bar{z}) = X^\mu(f(z), f(z)^*) + \frac{\alpha'}{2} V^\mu \log |f'(z)|^2. \quad (2.5)$$

In order for this symmetry to be compatible with the boundary conditions we shall require $v(z)^* = v(\bar{z})$, or equivalently $f(z)^* = f(\bar{z})$ for the finite transformation. The boundary is mapped into itself and the boundary fields $X^\mu(y)$, $y \in \mathbb{R}$ transform as

$$X^\mu(y) \rightarrow f \circ X^\mu(y) = X^\mu(f(y)) + \alpha' V^\mu \log |f'(y)|. \quad (2.6)$$

At the quantum level the theory remains conformal. The energy momentum tensor can be in the usual fashion expanded into the Virasoro modes

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} : \alpha_{m-n}^\mu \alpha_{n\mu} : + i \left(\frac{\alpha'}{2} \right)^{1/2} (m+1) V^\mu \alpha_{m\mu} \quad (2.7)$$

which form a Virasoro algebra with a central charge

$$c = D + 6\alpha' V_\mu V^\mu. \quad (2.8)$$

From the operator product expansion with $T(z)$ one gets the conformal weights of all primary operators, for example the weight of $e^{ikX(z, \bar{z})}$ is

$$\alpha' \left(\frac{k^2}{4} + i \frac{V^\mu k_\mu}{2} \right). \quad (2.9)$$

On the boundary however, the boundary normal ordered operator ${}^* e^{ikX(y)} {}^*$ has conformal weight

$$h = \alpha' (k^2 + ik_\mu V^\mu) \quad (2.10)$$

by virtue of the mirror image term in the free field propagator of X^μ .

For applications to string field theory we need also the correlators of the theory. The simplest boundary correlators

$$\langle e^{ik_1 X(y_1)} e^{ik_2 X(y_2)} \dots e^{ik_n X(y_n)} \rangle = i C_{D_2}^X (2\pi)^d \delta \left(\sum_{i=1}^n k_i^\mu + i V^\mu \right) \prod_{1 \leq i < j \leq n} |y_i - y_j|^{2\alpha' k_i k_j} \quad (2.11)$$

differ from the corresponding correlators in the constant dilaton background only through the modified momentum conservation law which comes from the integration over the zero modes of X^μ . This is the *only* way in which the amplitude is affected by the non-constant dilaton. The delta function of a complex argument is a somewhat formal object which should be thought of in the position space, in terms of the integral representation of the delta function. That is, define

$$\delta(k + iV) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx e^{ikx - Vx} \quad (2.12)$$

where x is the zero mode of the field X . This object is, of course, divergent. In all quantities of interest to us, the divergence will be irrelevant. We note that all formulæ involving the modified delta function can be obtained also via conformal perturbation theory in the dilaton gradient.

3 OSFT in the linear dilaton background

To study open string field theory on a space-filling D-brane in the linear dilaton background, one does not need much.¹ The string field is an element of the Hilbert space of the linear dilaton CFT which can be built upon a $SL(2, \mathbb{R})$ invariant vacuum, that we are going to call $|0\rangle$. One can use the 26 free worldsheet bosons quantized in the usual way, together with the b, c reparametrization ghosts to create Fock space identical to the one with constant dilaton, up to the (un)important distinction of which states are called normalizable and which non-normalizable.² The action of open string field theory

$$S_{OSFT} = -\frac{1}{g_o^2} \left[\frac{1}{2} \langle \Psi, Q_B \Psi \rangle + \frac{1}{3} \langle \Psi, \Psi * \Psi \rangle \right] \quad (3.1)$$

thus receives the information about the linear dilaton from two sources. First are the conformal properties of the vertex operators, which are important both for the action of the BRST charge Q_B , and for the conformal mappings used to define the cubic vertex. The second place where the linear dilaton background enters is the nontrivial background charge in the disk correlation functions.

It is instructive to see how all this works in level truncation. Let us truncate the whole string field to the tachyon

$$\Psi = t(X) c_1 |0\rangle. \quad (3.2)$$

The vertex operator $t(X)$ can be thought of as a *formal* superposition of vertex operators

$$t(X) = \int \frac{d^D k}{(2\pi)^D} t(k) e^{ikX}. \quad (3.3)$$

The function $t(x)$ has the interpretation as the classical x^μ -dependent tachyon field.

Inserting the truncated string field (3.3) into the action (3.1) one finds³

$$S_{tachyon} = -\frac{1}{g_o^2} \int d^D x e^{-V_\mu x^\mu} \left[\frac{1}{2} \alpha' (\partial t)^2 - \frac{1}{2} t^2 + \frac{1}{3} K^{-3+\alpha' V^2} t^3 \right]. \quad (3.4)$$

The notorious constant K is

$$K = \frac{4}{3\sqrt{3}} \approx 0.7698 \quad (3.5)$$

¹Lower dimensional D-branes can be treated analogously as long as the dilaton gradient points along the worldvolume. If the dilaton gradient were misaligned, there would be a net force on the D-brane and the transverse scalars would become time dependent.

²In string field theory, even in the flat background with constant dilaton, the issue of normalizability is rather subtle and not well understood. The linear dilaton background introduces an additional layer of complexity to the question. We shall ignore both issues and leave up to the reader and not some ad hoc norm to judge the physical relevance of our solutions.

³Detailed computation of this type can be found i.e. in [17].

and \tilde{t} is defined by

$$\tilde{t} = K^{-\alpha'} \square t. \quad (3.6)$$

The appearance of the V^2 term in the exponent in front of the cubic term might, at first sight, seem quite surprising. It looks as if the coefficients of the true vacuum state were to depend on the dilaton gradient which would contradict Sen's result [2]. Happily, the coefficient is just right, so that when deriving the equations of motion first, and restricting to space-time constant solutions afterwards, the V^2 -dependence completely drops out! It is only the total energy that depends on the dilaton gradient. Even that dependence is very simple: the energy density of the vacuum is proportional to the quantity e^{-Vx} , which integrates to a divergent quantity in the weak-coupling region.

There is one more point we wish to make about the tachyon coupling to the dilaton. Naively, one could have expected a coupling of the form $e^{-\Phi} t \square t$. There are at least three heuristic arguments why it should be so. First, the $\langle \Psi, Q_B \Psi \rangle$ contains manifestly p^2 inside Q_B , although more careful computation shows, that it is the conformal weight of e^{ikX} that matters, and this contains a linear term in the momentum. Second, a possible, from certain points of view quite natural, field redefinition (3.6), would introduce such couplings in the kinetic term. The third, and perhaps the strongest arguments, are based on intuition coming from supersymmetric theories. As discussed from various perspectives in [32], the half-BPS operator in $N = 4$ that couples to spatially varying dilaton is the Lagrangian whose scalar field part has to be written as $-\text{Tr } X^i \square X^i$. How do we reconcile this intuition with our OSFT computation? While we have not done a rigorous computation from first principles, we believe that the resolution to this paradox is that the generic kinetic-term couplings of scalars to the dilaton are of the form

$$\int d^D x e^{-\Phi} \left(\partial_\mu X \partial^\mu X + \frac{1}{2} (\square \Phi) X^2 \right). \quad (3.7)$$

In the linear dilaton background the second term would be zero, and the coupling agrees nicely with what we find from string field theory. Varying with respect to infinitesimal dilaton, on the other hand, and setting it to some constant value Φ_0 would give a coupling

$$\int d^D x \delta \Phi e^{-\Phi_0} (X \square X) \quad (3.8)$$

which looks as if it came from $\int e^{-\Phi} (-X \square X)$ in accord with supersymmetry arguments.

To better understand the nature of the OSFT in the dilaton background, let us compute the quadratic part of the action for level zero fields: the gauge field A_μ and the Nakanishi-Lautrup field β .

$$\begin{aligned} \Psi &= \int \frac{d^D k}{(2\pi)^D} [A_\mu(k) \alpha_{-1}^\mu c_1 + \beta(k) c_0] e^{ikX} |0\rangle \\ &= \int \frac{d^D k}{(2\pi)^D} \left[A_\mu(k) \frac{i}{\sqrt{2\alpha'}} c \partial X^\mu(0) + \beta(k) \partial c(0) \right] e^{ikX(0)} |0\rangle \end{aligned} \quad (3.9)$$

We do not use the doubling trick for the matter operators inserted on the boundary.⁴

The quadratic part of the action can be found to be

$$S_{A+\beta}^{quadr} = -\frac{\alpha'}{2g_o^2} \int e^{-Vx} \left[\partial_\mu A_\nu \partial^\mu A^\nu - 2\alpha' \beta^2 + \sqrt{2\alpha'} i A_\mu \partial^\mu \beta \right]. \quad (3.10)$$

Integrating out β via its equation of motion one finds that various V^μ dependent terms arising at intermediate stages all cancel out and one is left with

$$S_{A,eff}^{quadr} = -\frac{\alpha'}{4g_o^2} \int e^{-Vx} F_{\mu\nu} F^{\mu\nu}. \quad (3.11)$$

It could be interesting to compute the cubic and the effective quartic term in the non-abelian setup along the lines of [18], but we will not attempt it here.

It is tempting to guess that a general form of the OSFT action written in modes in the linear dilaton background will have the general form

$$S_{LD OSFT} = \frac{1}{g_o^2} \int e^{-Vx} [\mathcal{L}_{V=0} + \partial Z], \quad (3.12)$$

where Z is some function of the fields and their derivatives (with the typical OSFT nonlocalities) that does not depend on the dilaton gradient V^μ . This guess is actually true for an elementary reason. The momentum conservation delta function gives the exponential factor for both the quadratic and cubic terms. The only other influence the dilaton background has is through modified conformal (or BRST) transformation properties which are analytic in V^μ . Therefore $\mathcal{L}_V = \mathcal{L}_{V=0} + V^\mu \mathcal{L}_\mu^{(1)} + O(V^2)$. All V 's can then be replaced by total derivatives because of the exponential prefactor.

4 Light-like tachyon rolling in OSFT

In this section we are going to construct some new exact rolling-tachyon solutions of OSFT, in which the fields depend only on a single light-like coordinate. Introducing new light-cone coordinates

$$X^\pm = \frac{1}{\sqrt{2}} (X^0 \pm X^1), \quad (4.1)$$

in which the space-time metric becomes $ds^2 = -dt^2 + dx_1^2 + \dots + dx_p^2 = -2dx^+ dx^- + dx_2^2 + \dots + dx_p^2$, we may ask for solutions depending on, say, X^+ only. So far, most of the work in the literature concerned homogenous rolling tachyon decay, where the tachyon field depends on the time-like X^0 coordinate only. Our motivation for the light-like case is twofold: First, as we shall see X^+ dependence is much easier to deal with analytically and allows for constructing exact solution.

⁴The matter operator $X(y)$ is treated as a boundary operator, and the derivative in $\partial X(y)$ is with respect to the boundary coordinate y . This explains perhaps unfamiliar looking factor in (3.9).

Second, it looks very likely to us that solutions depending on the light-cone coordinates are in fact more physically relevant ones. One would expect that any local perturbation that destabilizes an unstable D-brane (or brane-antibrane pair) should start spreading out with a speed of light, eating holes in the D-brane worldvolume. At large distances from the original seed of instability the spherical tachyon wave looks locally like a flat plane; this is the limit that our solutions shall describe.

4.1 Level zero truncation

It is instructive to first understand such light-like rolling tachyons qualitatively, at the lowest truncation level of open string field theory. For reasons that will become clear soon we turn on arbitrary nonzero dilaton gradient with $V^+ > 0$. The equation of motion from (3.4) reads

$$\alpha' \partial \left(e^{-Vx} \partial t(x) \right) + e^{-Vx} t(x) - K^{-3+\alpha'V^2-\alpha'\square} (e^{-Vx} \tilde{t}^2) = 0. \quad (4.2)$$

Imposing the light-like ansatz $t(x) = t(x^+)$ we find $t(x^+) = \tilde{t}(x^+)$, and the equation of motion takes particularly simple form

$$(\alpha' V^+ \partial_+ - 1) t(x^+) + K^{-3} [t(x^+ + 2\alpha' V^+ \log K)]^2 = 0. \quad (4.3)$$

This equation can be solved order by order in an expansion

$$t(x^+) = K^3 \sum_{n=1}^{\infty} a_n \exp \left(\frac{nx^+}{\alpha' V^+} \right), \quad (4.4)$$

where the coefficients a_n are determined by a simple recursion relation:

$$a_n = -\frac{K^{2n}}{n-1} \sum_{k=1}^{n-1} a_k a_{n-k}. \quad (4.5)$$

One can guess – and justify *a posteriori* – that the leading contribution from the sum comes from $k \approx n/2$ and therefore the coefficients behave as

$$a_n \approx K^{2n \log_2 n}. \quad (4.6)$$

Since $K < 1$ the series (4.4) has an infinite radius of convergence and can thus be straightforwardly evaluated numerically, see Fig. 1.

We find that the solution interpolates in x^+ (or in time x^0 if we fix x^1) between the perturbative D-brane vacuum in the far past, and the true closed string vacuum in the far future, rather than oscillating wildly. Given our experience with time-like solutions, this is most unexpected!

As is clear from the graph, the solution is approaching the closed string vacuum, but not monotonically. The exact form of the oscillations for large x^+ is easy to derive by linearizing the equation (4.3) around the vacuum $t = t_* + \delta t$. In units $\alpha' V^+ = 1$, the solution is

$$\delta t = A e^{\omega x^+} + A^* e^{\omega^* x^+}, \quad (4.7)$$

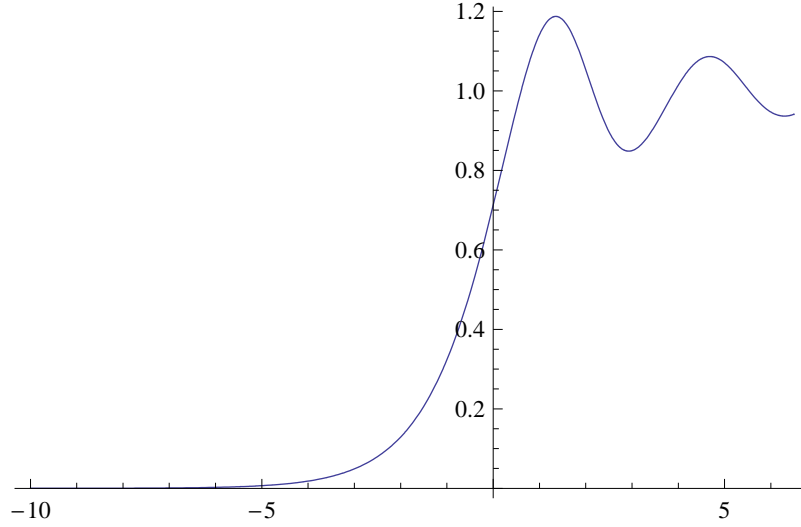


Figure 1: The plot of series (4.4) with initial condition $a_1 = 1$ in units $\alpha'V^+ = 1$. The whole series was normalized by K^{-3} so that the tachyon vacuum (at level zero truncation) appears at $t = 1$. This plot was obtained by summing first 5000 terms with high enough accuracy of 500 digits.

where the constant ω obeys a transcendental equation

$$\omega - 1 + 2K^{2\omega} = 0. \quad (4.8)$$

It has an infinite number of solutions, and they can be found numerically. The smallest one in the absolute value is $\omega = -0.23797 \pm 1.89699i$. Note that $|\omega| = 1.91186$ is significantly larger than the 1 of the tachyon in the perturbative vacuum. This is a level-zero manifestation of the absence of open string modes in the tachyon vacuum [56, 57, 58, 59, 60, 61].

Before we move on to the full-fledged string field theory, let us pause to explain the role of the linear dilaton background. From the equation (4.3) together with Fig. 1, we see that the tachyon changes from the perturbative vacuum value ($t = 0$) to the true vacuum value ($t = t_*$) very sharply, on length scale of order $\alpha'V^+$. Thus, in the limit of vanishing dilaton gradient, the tachyon jumps from the value $t = 0$ to t_* infinitely quickly.

4.2 Full exact solution

Rolling tachyon solutions, such as those discussed in the level-truncated theory in the previous subsection, can be easily constructed exactly in the full OSFT. Our particular light-like rolling tachyon solution is described by a free-boson conformal field theory deformed by an exactly marginal boundary operator $e^{\beta X^+}$, where $\beta = \frac{1}{\alpha'V^+}$, so that its boundary conformal dimension

in the linear dilaton background equals one. In string field theory such solutions can be obtained perturbatively in the marginal deformation parameter.⁵ Recently such solutions, exact to all orders, have been found [15, 16] in the very convenient \mathcal{B}_0 -gauge of [14].⁶ Generalizations and related work can be found in [15, 16, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50].

As in the previous subsection we take for definiteness $V^+ > 0$, so that $\beta > 0$, and the tachyon rolls from the perturbative open string vacuum in the past null infinity to the true vacuum in the future null infinity, as we shall see. An important property of $e^{\beta X^+}$ is that it has trivial Wick contractions with itself. One can therefore use the results of [15, 16] to write down open string field theory solution describing the deformed CFT. The solution takes the form

$$\Psi = \sum_{n=1}^{\infty} \lambda^n \left(-\frac{\pi}{2}\right)^{n-1} \int_0^1 \int_0^1 \dots \int_0^1 \prod_{i=1}^{n-1} dr_i \mathcal{F} \left(\sum_{k=1}^{n-1} r_k \right) e^{\beta X^+(x_1)} e^{\beta X^+(x_2)} \dots e^{\beta X^+(x_n)} |0\rangle, \quad (4.9)$$

where

$$\begin{aligned} \mathcal{F}(r) &= e^{-\frac{r}{2}\hat{\mathcal{L}}} \left[-\frac{1}{\pi} \hat{\mathcal{B}}c \left(\frac{\pi}{4}r \right) c \left(-\frac{\pi}{4}r \right) + \frac{1}{2} \left(c \left(\frac{\pi}{4}r \right) + c \left(-\frac{\pi}{4}r \right) \right) \right] \\ &= \sum_{k=0}^{\infty} \mathcal{F}_L r^L. \end{aligned} \quad (4.10)$$

The operator-valued coefficients \mathcal{F}_L in the Taylor expansion of \mathcal{F} are eigenstates of \mathcal{L}_0 with eigenvalue $L - 1$. The first few are $\mathcal{F}_0 = c$, $\mathcal{F}_1 = -\frac{1}{2}\hat{\mathcal{L}}c + \frac{1}{2}\hat{\mathcal{B}}c\partial c, \dots$. As explained in [15, 16] the insertion points x_j are functions of r_k

$$x_i = \frac{\pi}{4} \left(\sum_{k=1}^{n-1} r_k - 2 \sum_{k=1}^{i-1} r_k \right). \quad (4.11)$$

Let us write the string field in the form

$$\Psi = \sum_{L=0}^{\infty} \mathcal{F}_L f^{(L)}(X^+, \partial X^+, \dots) |0\rangle. \quad (4.12)$$

The operator coefficients $f^{(L)}$ can further be decomposed as

$$\begin{aligned} f^{(L)}(X^+, \partial X^+, \dots) &= f_1^{(L)}(X^+) + f_{\partial X^+}^{(L)}(X^+) \partial X^+ + f_{(\partial X^+)^2}^{(L)}(X^+) (\partial X^+)^2 \\ &\quad + f_{\partial^2 X^+}^{(L)}(X^+) \partial^2 X^+ + \dots \end{aligned} \quad (4.13)$$

The coefficients $f_{\partial^m X^+ \dots}^{(L)}(X^+)$ are operators from the viewpoint of conformal field theory, but can be regarded as X^+ dependent wave functions in string field theory. They are given by infinite sums of exponentials $e^{n\beta X^+}$.

⁵For nonperturbative approach to marginal deformations see [35].

⁶Other gauges for OSFT were recently considered in [51, 52, 53, 54, 55].

To evaluate the operators $f_{\dots}^{(L)}(X^+)$ we first have to develop the operator product expansion

$$\begin{aligned} e^{\beta X^+(x_1)} e^{\beta X^+(x_2)} \dots e^{\beta X^+(x_n)} &= e^{n\beta X^+} e^{\beta \sum_{p=1}^{\infty} \frac{1}{p!} (\sum_{j=1}^n x_j^p) \partial^p X^+(0)} \\ &= e^{n\beta X^+} \left[1 + \beta \left(\sum_{j=1}^n x_j \right) \partial X^+ + \frac{\beta^2}{2} \left(\sum_{j=1}^n x_j \right)^2 (\partial X^+)^2 + \frac{\beta}{2} \left(\sum_{j=1}^n x_j^2 \right) (\partial^2 X^+) + \dots \right] \end{aligned} \quad (4.14)$$

to the desired order. The crucial simplification is that we do not need to worry about normal-ordering: Any product of operators made only from the free field X^+ does not need normal ordering, since X^+ has vanishing Wick self-contractions.

Next step in evaluating the coefficients $f_{\dots}^{(L)}(X^+)$ is to compute the $n-1$ dimensional integral over r_k of $(\sum r_k)^L$ times a polynomial in the x_j . The polynomials that appear are of the form

$$\left(\sum_{j=1}^n x_j^{p_1} \right) \left(\sum_{j=1}^n x_j^{p_2} \right) \dots \quad (4.15)$$

and they have to be first re-expressed in terms of r_k 's. This way we get similar structure but with coefficients which are in general polynomial functions of n . The first two relevant relations of this sort are

$$\begin{aligned} \sum_{j=1}^n x_j &= \frac{\pi}{4} \sum_{k=1}^{n-1} (2k-n) r_k, \\ \sum_{j=1}^n x_j^2 &= \left(\frac{\pi}{4} \right)^2 \left[n \sum_{k=1}^{n-1} r_k^2 + \sum_{1 \leq k < l \leq n-1} (2n-4(l-k)) r_k r_l \right]. \end{aligned} \quad (4.16)$$

The actual integration is best performed by writing

$$\left(\sum r \right)^L = \frac{d^L}{d\alpha^L} e^{\alpha \sum r} \Big|_{\alpha=0} \quad (4.17)$$

and postponing the differentiation with respect to α to the very end. The multi-dimensional integral factorizes into $n-1$ one-dimensional integrals of simple exponentials. We easily find:

$$\begin{aligned} \int_0^1 \prod_{i=1}^{n-1} dr_i \left(\sum_{k=1}^{n-1} r_k \right)^L &= \frac{d^L}{d\alpha^L} \left(\frac{e^\alpha - 1}{\alpha} \right)^{n-1} \Big|_{\alpha=0} \\ \int_0^1 \prod_{i=1}^{n-1} dr_i \left(\sum_{k=1}^{n-1} r_k \right)^L r_k^2 &= \frac{d^L}{d\alpha^L} \left[\left(\frac{e^\alpha - 1}{\alpha} \right)^{n-2} \left(\frac{(\alpha-1)^2 e^\alpha + e^\alpha - 2}{\alpha^3} \right) \right] \Big|_{\alpha=0} \\ \int_0^1 \prod_{i=1}^{n-1} dr_i \left(\sum_{k=1}^{n-1} r_k \right)^L r_k r_l &= \frac{d^L}{d\alpha^L} \left[\left(\frac{e^\alpha - 1}{\alpha} \right)^{n-3} \left(\frac{(\alpha-1)e^\alpha + 1}{\alpha^2} \right)^2 \right] \Big|_{\alpha=0} \end{aligned} \quad (4.18)$$

Another useful relation, following from the permutation symmetry among the r_k 's and formula (4.16), is

$$\int_0^1 \prod_{i=1}^{n-1} dr_i \left(\sum_{k=1}^{n-1} r_k \right)^L \left(\sum_{j=1}^n x_j \right) = 0. \quad (4.19)$$

Finally, we sum over n to find a geometric series, and we readily obtain

$$\begin{aligned} f_1^{(L)}(X^+) &= \frac{d^L}{d\alpha^L} \frac{\lambda e^{\beta X^+}}{1 + \frac{\pi}{2} \lambda e^{\beta X^+} \frac{e^\alpha - 1}{\alpha}} \Big|_{\alpha=0} \\ &= \frac{2}{\pi} B_L + \left(\frac{2}{\pi} \right)^2 ((L-1)B_L + LB_{L-1}) \lambda^{-1} e^{-\beta X^+} + \dots, \end{aligned} \quad (4.20)$$

where B_L are the Bernoulli numbers. Note that the fortunate feature of finding geometric series allowed us to re-expand the original series in $e^{\beta X^+}$ in terms of $e^{-\beta X^+}$. The dots in the second line refer to subleading terms of the order $e^{-2\beta X^+}$ and higher, that we will not display explicitly. For the other coefficients we find similarly

$$f_{\partial X^+}^{(L)}(X^+) = 0, \quad (4.21)$$

$$\begin{aligned} f_{(\partial X^+)^2}^{(L)}(X^+) &= \frac{\beta^2}{4} \frac{d^L}{d\alpha^L} \left[\frac{(e^\alpha - 1)^2 - \alpha^2 e^\alpha}{((e^\alpha - 1)\pi \lambda e^{\beta X^+} + 2\alpha)^4} \right] \Big|_{\alpha=0} \pi^4 \lambda^3 e^{3\beta X^+} \\ &= \frac{\beta^2}{4} \left(\frac{L-3}{6} B_{L+2} + \frac{L-2}{2} B_{L+1} + \frac{L-1}{3} B_L \right) \lambda^{-1} e^{-\beta X^+} + \dots, \end{aligned} \quad (4.22)$$

$$\begin{aligned} f_{\partial^2 X^+}^{(L)}(X^+) &= -\frac{\beta}{4} \frac{d^L}{d\alpha^L} \left[\frac{((e^\alpha - 1)^2 - \alpha^2 e^\alpha) \pi \lambda e^{\beta X^+} + (6\alpha - 4\alpha^2) e^\alpha - 4\alpha}{((e^\alpha - 1)\pi \lambda e^{\beta X^+} + 2\alpha)^4} \right] \Big|_{\alpha=0} \pi^3 \lambda^2 e^{2\beta X^+} \\ &= -\frac{\beta}{4} \left(\frac{L-3}{6} B_{L+2} + \frac{L-2}{2} B_{L+1} + \frac{L-1}{3} B_L \right) \lambda^{-1} e^{-\beta X^+} + \dots, \end{aligned} \quad (4.23)$$

The large- x^+ -behavior of these three wave functions above clearly illustrates the validity of our central result, which we first asserted in the introduction: *in the limit of large λ , or equivalently large x^+ the solution limits to the tachyon vacuum constructed in [14]*. We will now go on to prove that our assertion is valid for all components of the string field.

4.3 Proof that the late time asymptotics is the tachyon vacuum

We start by observing that the generic coefficient $f_{\partial^{p_1} X^+ \partial^{p_2} X^+ \dots}^{(L)}(X^+)$ receives contribution solely from terms of the form proportional to

$$\sum_{n=1}^{\infty} \lambda^n e^{n\beta X^+} \left(-\frac{\pi}{2} \right)^{n-1} \int_0^1 \int_0^1 \dots \int_0^1 \prod_{i=1}^{n-1} dr_i \left(\sum_{k=1}^{n-1} r_k \right)^L \left(\sum_{j_1=1}^n x_{j_1}^{p_1} \right) \left(\sum_{j_2=1}^n x_{j_2}^{p_2} \right) \dots \quad (4.24)$$

up to simple overall combinatorial factors depending on p 's. The $n - 1$ -tuple integral turns out to be a polynomial in n of order $L + \sum(p_m + 1)$. To see the order of the polynomial, note that n appears only through upper limits on the sums. The integration itself does not give rise to any n dependence.⁷ One could think of adding an arbitrary number of integrations over r_n, \dots, r_{N-1} . Since the integrand does not depend on those variables, this will simply contribute a factor of 1. Another way to say that is that the $n - 1$ -tuple integral of *e.g.* $r_k^\alpha r_l^\beta$ does not depend on n but only on α, β , and on whether or not k is equal to l . Reexpressing the x_j in terms of the r 's, we see that $\sum_{j=1}^n x_j^p$ is given by a $p + 1$ -fold sum over the r 's. Performing the integrations of the summands we end up eventually with $L + \sum(p_m + 1)$ -fold sum of a number independent of n . As a result, we get polynomial in n of order $L + \sum(p_m + 1)$.

The coefficients f are given by power series in $\lambda e^{\beta X^+}$ with polynomial coefficients. The resulting series

$$f(q) \equiv \sum_{n=1}^{\infty} P(n)q^n = \sum_{n=1}^{\infty} (a_k n^k + \dots a_1 n + a_0)q^n \quad (4.25)$$

has a *finite* radius of convergence. Thankfully, the sum over n can be performed exactly and always leads to a meromorphic function of q , which can be continued analytically beyond its unit radius of convergence. The asymptotic expansion near infinity can be easily obtained from⁸

$$f(q) = P\left(q \frac{d}{dq}\right) \frac{q}{1-q} = - \sum_{n=0}^{\infty} P(-n)q^{-n}. \quad (4.26)$$

In particular we see that the limit for q going to infinity is given by $-P(0) = -a_0$.

We wish to prove that in the limit of large λ , or equivalently large x^+ , the solution limits to the tachyon vacuum constructed in [14]. In order to do this, we have to show that $P(n = 0) = 0$ for all terms except those without any factor of $\sum_{j=1}^n x_j^p$ in the integrand. For such terms we have already proved the correct limit in the preceding subsection. As we have argued above, the result of the integral is a $L + \sum(p_m + 1)$ -fold sum of an integral over product of powers of r that is independent of n . Apart from their dependence on the powers of the r 's, the integrals also depend on whether the indices on the r 's are the same or different. That can be accounted for by the Kronecker deltas which in turn split the $L + \sum(p_m + 1)$ -fold sum into many nested sums, where summation limits of the inner sums depend on the summation index of the outer sums. Since the inner most summands are polynomials in the summation index and

⁷For more general dynamical solutions of OSFT structure of the term (4.24) is different, and the integral itself is no longer independent of n . In the particular case of the rolling tachyon with dependence on a time-like coordinate, the integrand in the \mathcal{L}_0 basis would contain an additional factor $\prod_{1 \leq i < j \leq n-1} (x_i - x_j)^2$. Integrating even a constant would then lead to an n -dependent result. We will have more to say on this issue in Sec. 7.

⁸Alternatively one can write the sum as $a_k Li_{-k}(q) + \dots a_0 Li_0(q)$. The polylogarithm of negative order can be expressed in terms of Eulerian numbers as $Li_{-m}(q) = (1-q)^{-m-1} \sum_{i=0}^m \langle m \rangle_i q^{m-i}$ and obviously goes to zero at infinity.

possibly n , their sum would be a polynomial in the summation limits and n . By induction this applies to all the sums, and the resulting expression is a polynomial in n only.

As long as there is a factor of the form $\sum_{j=1}^n x_j^p$ in the integrand, one can rewrite all the terms as an 'outermost' sum of the form $\sum_{j=1}^n$, with the summand being a polynomial in n and j . Thanks to the relation

$$\sum_{j=1}^n j^k = \sum_{m=0}^{k+1} \frac{B_m}{m!} \frac{k!}{(k-m+1)!} \left[(n+1)^{k-m+1} - 1 \right] , \quad (4.27)$$

which is valid for any integer $k \geq 0$, we see that the result of summation over j will be a polynomial in n that vanishes at $n = 0$. This concludes our proof.

To conclude this section let us make few comments. Somewhat surprisingly the solution relaxes to the tachyon vacuum too slowly, only exponentially. At face value, this could indicate existence of perturbative states around the tachyon vacuum, that were proved in [59] to be absent! In level zero truncated OSFT in Sec. 4.1 the tachyon field also relaxed to the vacuum exponentially, but with a somewhat higher exponent which was argued to be finite only as an artefact of level truncation. In the VSFT or p -adic models discussed later in Sec. 6 the decay to the vacuum is superexponential. How is it then possible that the string field approaches the tachyon vacuum as $e^{-\beta X^+}$? We believe that such perturbations of the vacuum do not constitute new states, but that they merely represent infinitesimal gauge transformations of the tachyon vacuum.

As we have seen above, the tachyon relaxes to the vacuum both in the level-zero truncation and also for the exact solution in the \mathcal{L}_0 basis. One is led to believe that the same would be true in the ordinary L_0 basis. It would be interesting to check this explicitly. If this works, this could be a strategy to find a solution for the true vacuum in the Siegel gauge. One would have to construct the analog of our light-like solution to all orders in the deformation parameter, re-expand for large X^+ and hopefully read off the Siegel gauge vacuum.

Last issue we wish to touch upon is the puzzling issue of the tachyon matter [68, 69]. Tachyon matter is a conjectured decay product of unstable D-brane in the form of a pressureless dust. It is quite non-trivial to compute the energy-momentum tensor for the exact solution. For one thing, in the \mathcal{L}_0 -basis all levels are mixed together in the Lagrangian and the energy-momentum tensor, which makes the problem harder to analyze. We shall instead compute the energy momentum tensor for some toy examples in Sec. 6. Those results do suggest the existence of the tachyon matter with the correct properties, although what pressureless dust means in the linear dilaton background is somewhat subtle. In Sec. 5 we will analyze the problem from the boundary state perspective.

5 Light-like tachyon rolling in the boundary state formalism

In this section we analyze the rolling of the open string tachyon from the formulation of string theory in terms of the worldsheet of the string. We will use the language of boundary states as well as the equivalent language in terms of states and operators in the open string channel where appropriate.

Following the methods used in [65] to study time-like tachyon condensation, we compute the sources \mathcal{B} and $\mathcal{A}^{\mu\nu}$ as functions of the spacetime coordinates x^\pm . These source functions are defined by the overlap of the boundary state with the vertex operator for a graviton or dilaton. Here x^+ will always be the direction on which the tachyon depends, and x^- will always be the complementary lightlike direction.

*Zero modes and constrained propagators*⁹

We must take exercise caution with the definition of any string theory quantity as a local function of spacetime, including the local stress tensor. The usual overlap between the boundary state and the graviton state is defined as an integral over all of spacetime, and does not by itself give a local stress tensor. In order to define a local stress tensor, we must fix the zero modes x^μ of the X^μ embedding coordinates in order to compute the stress-energy at a given point x^μ . The amplitude at fixed x^μ is not BRST invariant by itself, but the interaction of the brane stress-energy with the metric and dilaton are of course BRST-invariant when the closed string states are on-shell and the x^μ is integrated over.

So let us specify what we mean by the zero mode x^μ . We do *not* mean that we fix the value the X^μ field, averaged over the worldsheet. That is, we are not fixing $x_{\text{bulk}}^\mu \equiv \frac{1}{A_{ws}} \int_D d^2z X^\mu$. We mean, rather, that we fix the value of the zero mode x^μ as a degree of freedom in the boundary state wave-function. From the path integral point of view, this amounts to fixing

$$x^\mu \equiv x_{\text{boundary}}^\mu \equiv \frac{1}{2\pi} \int_{\partial D} |dz| X^\mu. \quad (5.1)$$

In terms of the path integral, the fixing of the boundary zero mode x^μ amounts to inserting a Lagrange multiplier into the worldsheet action, implementing a delta function constraint:

$$\delta \left(-x^\mu + \frac{1}{2\pi} \int_{\partial D} |dz| X^\mu \right) = \int d\Lambda \exp \left(2\pi i \Lambda_\mu \left[-x^\mu + \frac{1}{2\pi} \int_{\partial D} |dz| X^\mu \right] \right). \quad (5.2)$$

This effectively modifies the propagator for the X^μ fields. If the boundary condition of the unconstrained field X^μ is Neumann, the modified propagator is

$$\langle X^\mu(z, \bar{z}) X^\nu(w, \bar{w}) \rangle_x = x^\mu x^\nu + \eta^{\mu\nu} P(z, \bar{z}; w, \bar{w}), \quad (5.3)$$

⁹ Some material in this subsection uses a result from [34].

where

$$P(z, \bar{z}; w, \bar{w}) = -\frac{\alpha'}{2} [\ln |z - w|^2 + \ln |1 - z\bar{w}|^2] \quad (5.4)$$

on the unit disc. The brackets represent the expectation value as calculated in the *free* theory with the boundary average of X^μ set to x^μ . We will incorporate the effects of the tachyon perturbation by resumming conformal perturbation theory. Note that the modified propagator *does* satisfy the equation of motion in the bulk, but does *not* satisfy the Neumann condition on the boundary. This is as expected, since the Lagrange multiplier Λ_μ couples to the boundary only.

We also comment on the difference between bulk *vs.* boundary normal-ordering. The bulk normal-ordering on the unit disc is the one that takes no account of the boundary condition, and subtracts only the single logarithmic singularity. In other words, the boundary normal-ordered product between two X^μ fields subtracts the full propagator (5.4), whereas the bulk normal-ordering, denoted here by $: \quad :$, subtracts only the first of the two logarithms in (5.4). That is,

$$: X^\mu(z, \bar{z}) X^\nu(w, \bar{w}) : \equiv X^\mu(z, \bar{z}) X^\nu(w, \bar{w}) + \frac{\alpha'}{2} \eta^{\mu\nu} \ln |z - w|^2 \quad (5.5)$$

$$*_ X^\mu(z, \bar{z}) X^\nu(w, \bar{w})_* \equiv X^\mu(z, \bar{z}) X^\nu(w, \bar{w}) + \frac{\alpha'}{2} \eta^{\mu\nu} [\ln |z - w|^2 + \ln |1 - z\bar{w}|^2] \quad (5.6)$$

Following [65] we define a scalar and symmetric-tensor source function describing the brane. We do this by inserting graviton and dilaton vertex operators with zero momentum into the disc amplitude defined at fixed x^μ . There is some arbitrariness in this procedure, as the amplitude is not fully BRST-invariant with x^μ held fixed. For the purposes of this paper, any fixed- x^μ amplitude can be thought of as an auxiliary quantity that will ultimately be integrated over x^μ in order to generate BRST-invariant amplitudes.

We conclude the setup of this discussion by noting that our definition of the localized amplitude at x^μ is the one used implicitly in [65], though the explicit definition of x^μ as a boundary integral is not given in that paper.

The dilaton source function $\mathcal{B}(x^\mu)$

The dilaton couples to the worldsheet as $\int_{\partial D} \frac{K}{2\pi} \Phi(X) + \int_D \frac{\mathcal{R}}{4\pi} \Phi(X)$ in the action of the Euclidean path integral, where \mathcal{R} is the Ricci curvature of the intrinsic metric on the disc D and K is the extrinsic curvature of the boundary ∂D . We have chosen our intrinsic metric on the disc to be flat with unit radius. Therefore the intrinsic curvature \mathcal{R} vanishes, and the extrinsic curvature of the boundary is thus equal to 1. The integral of X^μ along the boundary is equal to $2\pi x^\mu$ by definition of x^μ , so the coupling of the dilaton to the worldsheet is simply $\exp(-\Phi(x))$, with $\Phi(x) = V_\mu x^\mu$.

The dilaton source at fixed x^μ , referred to in [65] as $\mathcal{B}(x)$, is calculated by computing the disc partition function at fixed x^μ , without insertions. We compute the form of $\mathcal{B}(x)$ by resumming conformal perturbation theory. That is, we compute free-field correlators of our vertex operators with n additional insertions of

$$\mathcal{O} \equiv \frac{\lambda}{2\pi} \int_{\partial D} |dz| \exp(\beta X^+(z, \bar{z})) \quad (5.7)$$

and then sum over all nonnegative values of n , with the factor $(-1)^n/n!$ that comes from expanding the exponentiation of the potential term. That is,

$$\mathcal{B}(x) \equiv \sum_0^\infty \frac{(-1)^n}{n!} \mathcal{B}_n(x) \quad (5.8)$$

$$\mathcal{B}_n(x) \equiv \langle \mathcal{O}^n \rangle_x, \quad (5.9)$$

where the correlator is evaluated in the pure linear dilaton background, and with the boundary average of X^μ constrained equal to x^μ .

The nonzero mode piece of the correlator $\mathcal{B}_n(x)$ is trivial; all Wick-contractions in the free field theory are proportional to $\eta^{\mu_1\mu_2}$, where μ_1 and μ_2 are coordinate indices of the fields being contracted. However, since $\mu_i = +$ for all fields in the tachyon perturbations, and since $\eta^{++} = 0$, the nonzero mode correlators simply vanish. The correlator $\mathcal{B}_n(x)$ is then given by its zero mode contribution,

$$\mathcal{B}_n(x) = \lambda^n \exp(-V \cdot x + n\beta x^+) \quad (5.10)$$

Resumming the series for \mathcal{B} we get a superexponential dependence of the tadpole on x^+ :

$$\mathcal{B}(x) = \exp(-V \cdot x - \lambda \exp(\beta x^+)) \quad (5.11)$$

Note that this falloff is much faster than the falloff in the case of time-like tachyon condensation in a static background.

Graviton vertex operator and metric source function $\mathcal{A}^{\mu\nu}$

The vertex operator for the graviton (omitting the ghost component) is given by

$$\mathcal{V}(z, \bar{z}) \equiv e_{\mu\nu} \mathcal{V}^{\mu\nu}(z, \bar{z}) \quad (5.12)$$

$$\mathcal{V}^{\mu\nu}(z, \bar{z}) \equiv : \exp(ik^\sigma X_\sigma) \partial X^\mu \bar{\partial} X^\nu : (z, \bar{z}) \quad (5.13)$$

where $e_{\mu\nu}$ satisfies the transversality condition $k^\mu e_{\mu\nu} = k^\nu e_{\mu\nu} = 0$. We take $k_\mu = 0$ which gives us

$$\mathcal{V}(z, \bar{z}) \equiv e_{\mu\nu} : \partial X^\mu \bar{\partial} X^\nu : (z, \bar{z}) \quad (5.14)$$

with $e_{\mu\nu}$ arbitrary.

Using

$$: \partial X^\mu(z) \bar{\partial} X^\nu(\bar{z}') := {}_\star \partial X^\mu(z) \bar{\partial} X^\nu(\bar{z}')_\star + \frac{\alpha'}{2(1 - z\bar{z}')^2} \eta^{\mu\nu} \quad (5.15)$$

we place the closed string vertex operator at the origin, so

$$\mathcal{V}(0) = e_{\mu\nu} \mathcal{V}^{\mu\nu}(0) \quad (5.16)$$

$$\mathcal{V}^{\mu\nu}(0) \equiv {}_\star \partial X^\mu(0) \bar{\partial} X^\nu(0)_\star + \frac{\alpha'}{2} \eta^{\mu\nu} . \quad (5.17)$$

We then define a metric source function $\mathcal{A}^{\mu\nu}(x)$, by calculating the disc partition function at fixed x^μ with $\mathcal{V}^{\mu\nu}(0)$ inserted. As before,

$$\mathcal{A}^{\mu\nu}(x) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathcal{A}_n^{\mu\nu}(x) \quad (5.18)$$

$$\mathcal{A}_n^{\mu\nu}(x) \equiv \langle \mathcal{O}^n \mathcal{V}^{\mu\nu}(0) \rangle_x \quad (5.19)$$

$$= \frac{\alpha'}{2} \eta^{\mu\nu} \mathcal{B}_n(x) + \langle \mathcal{O}^n {}_\star \partial X^\mu(0) \bar{\partial} X^\nu(0)_\star \rangle_x \quad (5.20)$$

There are several cases we can consider. First, we can consider the case in which the indices of the metric are transverse to the light-cone plane defined by the gradients of the tachyon and its complementary light-like direction. That is, if we take the tachyon gradient to lie in the x^+ direction, then $(\mu, \nu)(i, j)$, where $i, j = 2, \dots, D-1$.

Considering this case, the X^i fields have no Wick contractions with the $X^{0,1}$ fields, and their contraction with each other has been cancelled by the boundary normal ordering, so

$$\langle \mathcal{O}^n {}_\star \partial X^i(0) \bar{\partial} X^j(0)_\star \rangle_x = 0 , \quad (5.21)$$

which means $\mathcal{A}^{ij}(x) = +\frac{\alpha'}{2} \delta^{ij} \mathcal{B}(x)$.

The metric source function with one light-cone and one transverse index vanishes, since every order of conformal perturbation theory has one uncontracted field ∂X^i or $\bar{\partial} X^i$ in the expectation value.

When $\mu = \nu = +$, the entire amplitude vanishes, since there is a ∂X^+ in the correlator for every n , which has no nonzero contractions with any other X^+ field. It has no one-point function, so the amplitude vanishes.

Similarly, the correlation function

$$\langle \mathcal{O}^n {}_\star \partial X^\pm(0) \bar{\partial} X^\mp(0)_\star \rangle_x = 0 , \quad (5.22)$$

as well. While the ∂X^- or $\bar{\partial} X^-$ field has a nonzero contraction with the X^+ 's in \mathcal{O}^n , the field ∂X^+ or $\bar{\partial} X^+$ cannot be contracted with any other field, and has no expectation value. So the only contribution to $\mathcal{A}^{\pm\mp}(x)$ is from the explicit normal ordering term in (5.17), giving

$$\mathcal{A}^{\pm\mp}(x) = -\frac{\alpha'}{2} \mathcal{B}(x) \quad (5.23)$$

The correlator for $\mu = \nu = -$ is a bit more nontrivial. In this case, there are indeed nonzero contributions at every order in n . Let us use the general free-field formula for the correlator of boundary-normal-ordered exponentials:

$$\left\langle \prod_{A=1}^N \star \exp \left(\gamma_{\mu A}^{(A)} X^{\mu A}(z_A, \bar{z}_A) \right) \star \right\rangle_x = \exp \left(-V_{\mu} x^{\mu} + \sum_{A=1}^N \gamma_{\mu}^{(A)} x^{\mu} \right) \prod_{1 \leq B < C \leq N} e^{\gamma_{\mu}^{(B)} \gamma^{\mu(C)} P(z_A, \bar{z}_A; z_B, \bar{z}_B)} . \quad (5.24)$$

Now set:

$$N = n + 2 \quad (5.25)$$

$$\gamma_{\mu}^{(A)} X^{\mu} = \beta X^{+}, \quad A = 3, \dots, n+2 \quad (5.26)$$

$$z_A = y_{A-2}, \quad A = 3, \dots, n+2 \quad (5.27)$$

$$\gamma_{\mu}^{(1)} X^{\mu} = \epsilon_1 X^{-} \quad (5.28)$$

$$\gamma_{\mu}^{(2)} X^{\mu} = \epsilon_2 X^{-} \quad (5.29)$$

Expanding to first order in ϵ_1 and ϵ_2 , we get

$$\begin{aligned} & \left\langle \prod_{j=1}^n \star \exp \left(\beta X^{+}(y_j, \bar{y}_j) \right) \star X^{-}(z_1, \bar{z}_1) X^{-}(z_2, \bar{z}_2) \right\rangle_x \\ &= \exp(-V \cdot x + n\beta x^{+}) \left[x^{-} - \beta \sum_{j=1}^n P(z_1, \bar{z}_1; y_j, \bar{y}_j) \right] \left[x^{-} - \beta \sum_{k=1}^n P(z_2, \bar{z}_2; y_k, \bar{y}_k) \right] \end{aligned} \quad (5.30)$$

Differentiating with respect to z_1 and \bar{z}_2 and setting $z_1 = z_2 = 0$, we find

$$\left\langle \prod_{j=1}^n \star \exp \left(\beta X^{+}(y_j, \bar{y}_j) \right) \star \star \partial X^{-}(0) \bar{\partial} X^{-}(0) \star \right\rangle_x \quad (5.31)$$

$$= \beta^2 \exp(-V \cdot x + n\beta x^{+}) \sum_{j,k=1}^n P_{,z}(z_1, \bar{z}_1; y_j, \bar{y}_j) P_{,\bar{z}_2}(z_2, \bar{z}_2; y_k, \bar{y}_k) \quad (5.32)$$

For general z and w ,

$$P_{,z}(z, w) = \left(-\frac{\alpha'}{2} \right) \left[\frac{1}{z-w} + \frac{\bar{w}}{\bar{w}z-1} \right] \quad (5.33)$$

so for $z = 0$ and $w = e^{it}$, we have

$$P_{,z}(z, \bar{z}; w, \bar{w}) = +\alpha' e^{-it} \quad (5.34)$$

$$P_{,\bar{z}}(z, \bar{z}; w, \bar{w}) = +\alpha' e^{+it}, \quad (5.35)$$

which means that

$$\left\langle \prod_{j=1}^n \star \exp(\beta X^+(y_j, \bar{y}_j)) \star \star \partial X^-(0) \bar{\partial} X^-(0) \star \right\rangle_x \quad (5.36)$$

$$= \beta^2 \alpha'^2 \exp(-V \cdot x + n\beta x^+) \sum_{j,k=1}^n e^{i(t_j - t_k)} \quad (5.37)$$

The integration regions for t_j and t_k are independent, so integrating over t_j and t_k yields zero unless $j = k$. This produces a factor of $n\lambda^n$, giving

$$\langle \mathcal{O}^{\star} \star \partial X^-(0) \partial X^-(0) \star \rangle_x = n\lambda^n \beta^2 \alpha'^2 \exp(-V \cdot x + n\beta x^+) \quad (5.38)$$

for a total of

$$\mathcal{A}^{--}(x) = -\beta^2 \alpha'^2 \lambda \exp(-V \cdot x + \beta x^+ - \lambda e^{\beta x^+}) \quad (5.39)$$

6 Light-like tachyon rolling in p -adic string theory

One does not always have the luxury of an exact solution for string field theory, and so one often turns to simpler toy models that are believed to capture the relevant physics. Although this is not necessarily the case here, it might still be useful to look at our light-like tachyon solution from a different perspective.

One popular toy-model is the p -adic string, that was at a time considered quite seriously [20, 21, 22]. It was shown more recently in [23, 24, 26, 29] that it captures a lot of interesting tachyon physics, despite its inherent limitations. Very different approach is taken by the so called vacuum string field theory (VSFT). This is a standard OSFT in which the string field gets expanded around the tachyon vacuum, and an additional guess is made for the effective form of the kinetic operator. It is interesting to note that truncating such a theory to level zero one gets an action that looks as a hybrid of two p -adic models: one with $p = 2$ and another one with $p = 27/16 = 1.6875$.

To start directly with the p -adic model for our linear dilaton background is quite challenging, since the general form of the tachyon - dilaton effective action is not known. We will therefore follow a different path. We start with the proximity to vacuum string field which specifies the dilaton-tachyon couplings uniquely. This then suggests a preferred form of the p -adic Lagrangian. More naive version of the action, the one we arrived at first, is mentioned in subsection 6.3, as its classical solutions display rather remarkable properties related to the discrete logistic equation.

6.1 VSFT

The basic idea of vacuum string field theory is to replace the BRST charge Q_B in the action(3.1) with a ghost number one object that acts as a derivative, squares to zero and has no cohomology. Several candidates have been proposed, but they all reduce to c_0 in the lowest truncation level.

The action takes the form

$$S_{tachyon}^{\text{VSFT}} = -\frac{1}{g_o^2} \int d^D x \, e^{-V_\mu x^\mu} \left[\frac{1}{2} t^2 + \frac{1}{3} K^{-3+\alpha' V^2} t^3 \right]. \quad (6.1)$$

Imposing again the light-like ansatz $t(x) = t(X^+)$ the equation of motion reduces to a simple functional equation

$$t(X^+) + K^{-3} [t(X^+ + 2\alpha' V^+ \log K)]^2 = 0 \quad (6.2)$$

that can be readily solved if one imposes that in the far past the solution was approximately constant, i.e. sitting at the top of the tachyon potential. The solution is

$$t = -K^3 e^{-e^{\beta'} X^+}, \quad (6.3)$$

where the constant $\beta' = \frac{1}{\alpha' V^+} \frac{\log 2}{\log K^{-2}} \approx 1.3247\beta$. The perturbations in the far past are given by the physical tachyon mode and should behave as $\text{const.} + e^{\beta X^+}$. Interestingly we see, that the vacuum string field theory at level zero predicts the mass of the perturbative open-string tachyon with an error of only 32%. We shall not study the VSFT model any further, instead we turn our attention to the p -adic string.

6.2 VSFT motivated linear-dilaton p -adic string

The exact effective action for the open p -adic string tachyon has been found to be

$$S = \frac{1}{g_p^2} \int d^d x \left[-\frac{1}{2} \phi p^{-\alpha' \square} \phi + \frac{1}{p+1} \phi^{p+1} \right]. \quad (6.4)$$

To the best of our knowledge, no one has studied how to couple the dilaton to this action.¹⁰ As we intend to use such an action as a toy model only, we are free to make some guesses and work out the implications. Perhaps the most naive attempt is to multiply the whole square bracket in the integrand with $e^{-\Phi}$, where $\Phi = V_\mu X^\mu$ is the linear dilaton background. This leads to an action with fairly interesting dynamics which we discuss a bit in section 6.3. For now, let us consider the second most obvious way of coupling the dilaton

$$S = \frac{1}{g_p^2} \int d^d x \, e^{-V_\mu X^\mu} \left[-\frac{1}{2} \left(p^{-\alpha' \square/2} \phi \right)^2 + \frac{1}{p+1} \phi^{p+1} \right]. \quad (6.5)$$

Upon a field redefinition $\phi \rightarrow p^{-\alpha' \square/2} \phi$ this looks almost as the VSFT at level zero, except that there is a slight mismatch between the nonlocality, which would match for $p = K^{-2} = 27/16$, and the power of $\tilde{\phi}$, which would match for $p = 2$.

The equation of motion with our light-like ansatz reads

$$\phi(X^+ + \alpha' V^+ \log p) = p^{\frac{\alpha'}{2} V^2} [\phi(X^+)]^p, \quad (6.6)$$

¹⁰Couplings of background magnetic field, or the NS-NS B-field have been proposed in [79, 80, 81].

whose solution is

$$\phi(X^+) = p^{-\frac{\alpha' V^2}{2(p-1)}} e^{-e^{\beta X^+}}. \quad (6.7)$$

The action (6.5) passes one non-trivial check (in addition to those that do not depend on the dilaton coupling) that it correctly predicts the mass of the open-string tachyon in the linear dilaton background. This is manifested by the fact that X^+ dependence enters through $e^{\beta X^+}$, where $\beta = 1/(\alpha' V^+)$.

We can use the action (6.5) to compute a stress tensor by covariantizing the derivatives, and varying the action with respect to the metric around a flat background. Treating carefully the boxes in the exponent, see e.g. [27], we arrive to

$$\begin{aligned} T_{\mu\nu} = & g_{\mu\nu} \left[e^{-\Phi} \left(\frac{1}{2} \left(p^{-\frac{\alpha'}{2}} \square \phi \right)^2 - \frac{1}{p+1} \phi^{p+1} \right) + k \nabla_\rho \int_0^1 dt e^{-kt \square} \left(e^{-\Phi} p^{-\frac{\alpha'}{2}} \square \phi \right) e^{-(1-t)k \square} \nabla^\rho \phi \right] \\ & - 2k \int_0^1 dt \nabla_{(\mu} e^{-kt \square} \left(e^{-\Phi} p^{-\frac{\alpha'}{2}} \square \phi \right) e^{-(1-t)k \square} \nabla_{\nu)} \phi, \end{aligned} \quad (6.8)$$

where $k = \frac{\alpha'}{2} \log p$. This energy-momentum tensor obeys conservation law $\partial^\mu T_{\mu\nu} = V_\nu \mathcal{L}$. For a solution dependent on x^+ the individual components simplify to

$$\begin{aligned} T_{ij} &= -g_{ij} \mathcal{L} - g_{ij} k V^+ \int_0^1 dt e^{-kt V^2} e^{-V_\mu x^\mu} \phi(x^+ + 2kt V^+) \partial_+ \phi(x^+) \\ T_{+i} &= k V_i \int_0^1 dt e^{-kt V^2} e^{-V_\mu x^\mu} \phi(x^+ + 2kt V^+) \partial_+ \phi(x^+) \\ T_{++} &= -2k \int_0^1 dt e^{-kt V^2} \partial_+ \left(e^{-V_\mu x^\mu} \phi(x^+ + 2kt V^+) \right) \partial_+ \phi(x^+) \\ T_{+-} &= \mathcal{L} \\ T_{--} &= T_{-i} = 0 \end{aligned} \quad (6.9)$$

The large x^+ asymptotics is given by

$$\begin{aligned} T_{ij} &= \gamma^2 e^{-V_\mu x^\mu} \left(g_{ij} e^{-2e^{\beta x^+}} \right) \\ T_{+i} &= \gamma^2 e^{-V_\mu x^\mu} \left(-\frac{1}{2} \alpha' \beta V_i e^{-2e^{\beta x^+}} \right) \\ T_{++} &= \gamma^2 e^{-V_\mu x^\mu} \left(-\alpha' \beta^2 e^{-2e^{\beta x^+} + \beta X^+} \right) \\ T_{+-} &= \gamma^2 e^{-V_\mu x^\mu} \left(-\frac{1}{2} e^{-2e^{\beta x^+}} \right) \\ T_{--} &= T_{-i} = 0, \end{aligned} \quad (6.10)$$

where $\gamma = p^{-\frac{\alpha' V^2}{2(p-1)}}$ is prefactor from (6.7). Note, that due to the x^+ translation invariance, we could have written solution with λ_{p-adic} multiplying every occurrence of $e^{\beta x^+}$.

Comparing with the metric source functions $A^{\mu\nu}(x)$ obtained from the boundary state computation of section 5, we find *exact* agreement of the dominant hyper-exponential piece, provided we identify $\lambda_{p\text{-adic}} = \frac{1}{2}\lambda_{CFT}$. The subleading pieces do not agree exactly. It is not clear that exact agreement should be expected. The minimal covariantization of the truncated string field theory contains ambiguities involving total derivatives in the action, giving rise to non-minimal higher-dimension terms in the stress tensor. Secondly, as discussed in sec. 5, the stress tensor at a point is not a truly gauge invariant object, as the insertions of off-shell closed string vertex operators do not preserve BRST invariance, nor does the procedure of restricting the boundary state zero modes x^μ to particular values. It is intriguing that the hyperexponential falloff of the stress tensor nonetheless seems to be robust. It would be interesting to see to what extent this falloff could be described in terms of gauge-invariant amplitudes.

6.3 Logistic linear-dilaton p -adic string

Another possibility how to couple dilaton to the tachyon of open p -adic string theory is via

$$S = \frac{1}{g_p^2} \int d^d x e^{-V_\mu X^\mu} \left[-\frac{1}{2} \phi p^{-\frac{1}{2}\square} \phi + \frac{1}{p+1} \phi^{p+1} \right]. \quad (6.11)$$

One could doubt whether such a coupling to the dilaton is realistic, but because in some sense this is the most naive coupling and also leads to most interesting dynamics, we will share some of it with the reader.

The equation of motion is

$$\phi(X^+) + p^{-\frac{1}{2}\alpha' V^2} \phi(X^+ + \alpha' \log p V^+) = 2\phi^p(X^+). \quad (6.12)$$

Interestingly, for $p = 2$, this is (a functional extension of) a discrete logistic map. Denoting $x_n = 2^{\frac{1}{p-1}} \phi(X^+ + n\alpha' \log p V^+)$ we find the generalized logistic map equation

$$x_{n+1} = r x_n (1 - x_n^{p-1}), \quad (6.13)$$

which reduces to the standard logistic equation for $p = 2$. The parameter r is given by

$$r = -p^{\frac{1}{2}\alpha' V^2}. \quad (6.14)$$

Usually the logistic map is considered only for $r > 0$. However, for $p = 2$ the logistic map has a simple self-duality property

$$r \rightarrow 2 - r \quad (6.15)$$

$$x_n \rightarrow \frac{1-r}{2-r} + \frac{r}{2-r} x_n, \quad (6.16)$$

so that negative values of r are meaningful as well and do not need any further analysis.

The functional equation (6.12) admits interesting exact solutions for a few special values of the parameters p and r that we shall mention later, but let us first discuss more general features of the equation.

The two stationary points¹¹ of the equation are given by $\phi = 0$ or $\phi = \phi_*$, where

$$\phi_* = \left(\frac{r-1}{2r} \right)^{\frac{1}{p-1}} = \left(\frac{1+p^{+\frac{1}{2}\alpha'V^2}}{2} \right)^{\frac{1}{p-1}}. \quad (6.17)$$

Zero is an attractive fixed point for $r \in (-1, 1)$, whereas ϕ_* is an attractive fixed point for $r \in (1, \frac{p+1}{p-1})$, assuming $p > 1$. The parameter r given by (6.14) is always negative and therefore the fixed point ϕ_* is never attractive. It can be identified with the unstable perturbative vacuum of the p -adic string. The dependence of ϕ_* on V^2 can be removed by a field redefinition. This will in turn produce similar factors in the action as in (3.4).

Solutions that start rolling downhill from ϕ_* in the far past can be expanded in exponentials

$$\phi(X^+) = \phi_* + \sum_{n=1}^{\infty} b_n \left(p^{+\frac{1}{2}\alpha'V^2} (p-1) + p \right)^{\frac{nX^+}{\alpha'(\log p)V^+}}. \quad (6.18)$$

The coefficients can be determined recursively for any p , but for $p = 2$ the result is especially simple

$$b_n = -\frac{2r}{(2-r)^n - (2-r)} \sum_{k=1}^{n-1} b_k b_{n-k}. \quad (6.19)$$

Their leading asymptotic behavior is $b_n \sim n^{-n \log_2(2-r)}$ and therefore the series (6.18) has infinite radius of convergence. To construct such solutions numerically for a given X^+ interval, one can in principle sum a sufficient number of terms with sufficient accuracy, but this quickly becomes rather time consuming. Smarter way is to start sufficiently close to the top where approximation $\phi \approx \phi_* + b_1 \left(p^{+\frac{1}{2}\alpha'V^2} (p-1) + p \right)^{\frac{X^+}{\alpha'(\log p)V^+}}$ is reliable over an interval of length $\alpha' \log p V^+$, and then use the functional relation (6.12) to replicate the function.

As we have mentioned, there are some special solutions that can be described analytically. In the case of $p = 2$ we can have space-like dilaton $\alpha'V^2 = 2$ corresponding to $r = -2$ for which there is a simple solution

$$\phi(X^+) = \frac{1}{4} + \cos \left(e^{\frac{X^+}{\alpha'V^+}} \right). \quad (6.20)$$

Such a dilaton background would exist in $D = 14$ for bosonic string or $D = 2$ for the superstring. It is interesting to see that for superstring the value of $\alpha'V^2 = 2$ is a critical value beyond which the usual notion of space-time does not make sense (as $D < 2$), and at the same time rolling process starts exhibiting divergences.

¹¹For p odd there is a third stationary point $-\phi_*$. It will not play a role in our discussion.

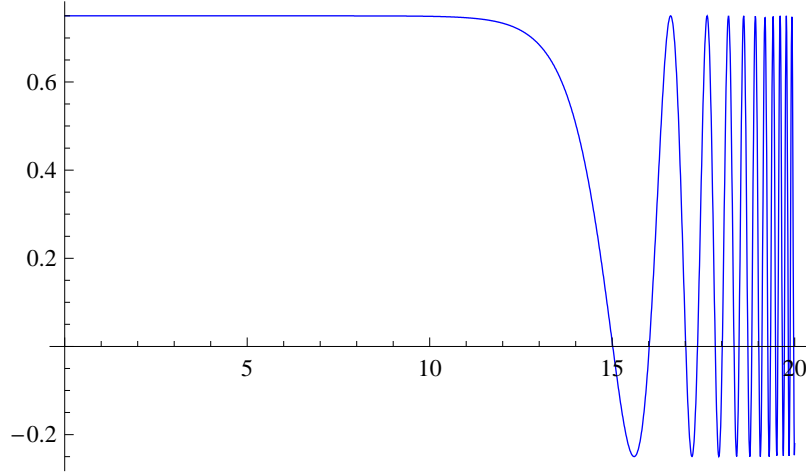


Figure 2: Rolling tachyon field in p -adic string theory with $p = 2$ and $r = -2$ given by (6.20). As an initial condition we have set $\phi = 1 - 10^{-9}$ at $X^+ = 0$. The value of r corresponds to the space-like dilaton with $\alpha'V^2 = 2$ that in the case of superstring gives maximally subcritical theory in $D = 2$.

Another case that allows a simple solution is $r = 0$ and any p . The solution

$$\phi(X^+) = \left(\frac{1}{2} p^{-\frac{1}{2} \alpha' V^2} \right)^{\frac{1}{p-1}} e^{-e^{\frac{X^+}{\alpha' V^+}}} \quad (6.21)$$

describes superfast decay to the true vacuum. Such a solution is relevant to the maximally supercritical theory with $V^2 \rightarrow -\infty$ or $D = \infty$.

The solution for a light-like dilaton $V^2 = 0$, or $r = -1$, has not been found in a closed form, but is well understood numerically; see Fig. 4. A slightly space-like linear dilaton (in the subcritical string) moves us beyond the first bifurcation in the logistic map and the rolling tachyon starts to develop interesting behavior; see Fig. 5. The tachyon field no longer relaxes to zero, but since the discrete system has length-two cyclic attractor, the function itself gradually develops step-like behavior.

As we increase the value of V^2 (or equivalently the value of $|r|$) we encounter a second bifurcation, see Fig. 6, beyond which a length-four cyclic attractor appears, see Fig. 7. Asymptotically the function approaches a step-like function which oscillates between four different values with a period equal to four in units of $\alpha' \log p V^+$.

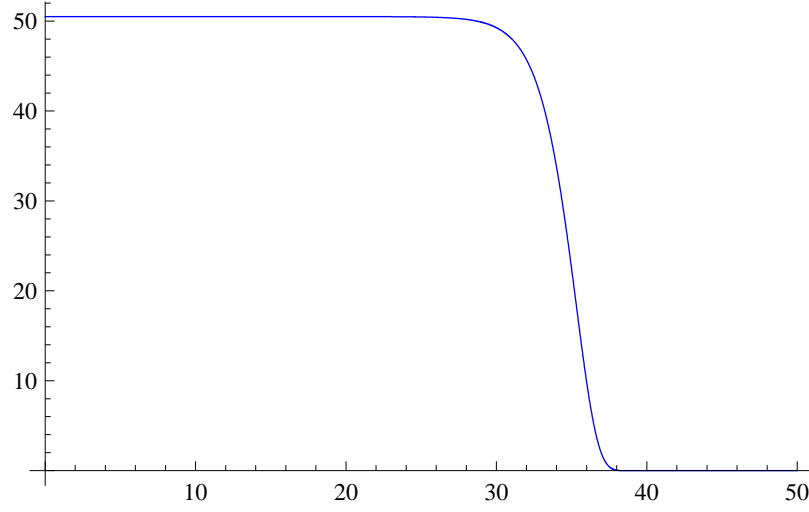


Figure 3: Rolling tachyon field in p -adic string with $p = 2$ and $r = -0.01$, which shows the behavior near $r = 0$. We see perfect agreement with (6.21). For the initial condition we have set $\phi = 1 - 10^{-9}$ at $X^+ = 0$. Such a solution describes maximally supercritical theory in $D = \infty$ with time-like dilaton with $V^2 \rightarrow -\infty$.

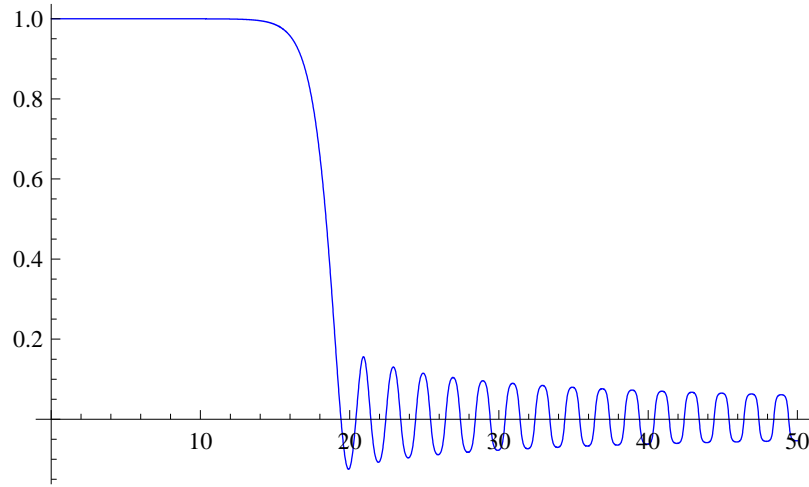


Figure 4: Rolling tachyon field in p -adic string, with $p = 2$, in the background of light-like linear dilaton $V^2 = 0$. The parameter $r = -1$, which is the well known point of the first bifurcation. The initial condition was set to $\phi = 1 - 10^{-9}$ at $X^+ = 0$

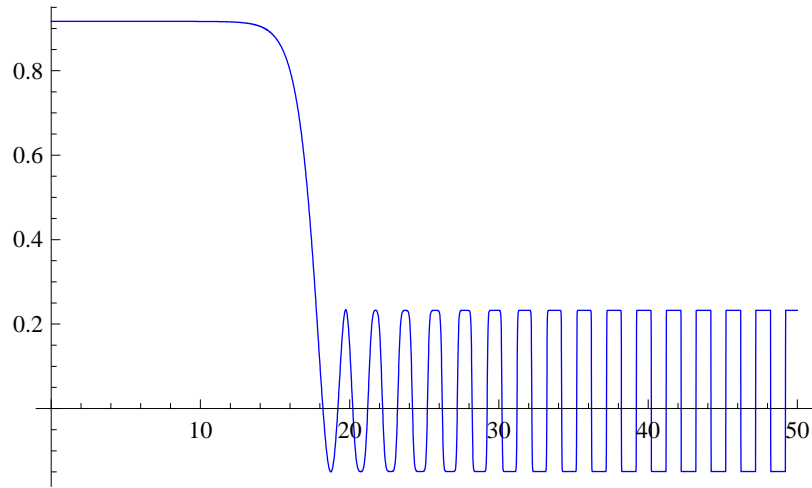


Figure 5: Evolution of the tachyon field for p -adic string with $p = 2$ and $r = -1.2$. For the initial condition we have set $\phi = 1 - 10^{-9}$ at $X^+ = 0$

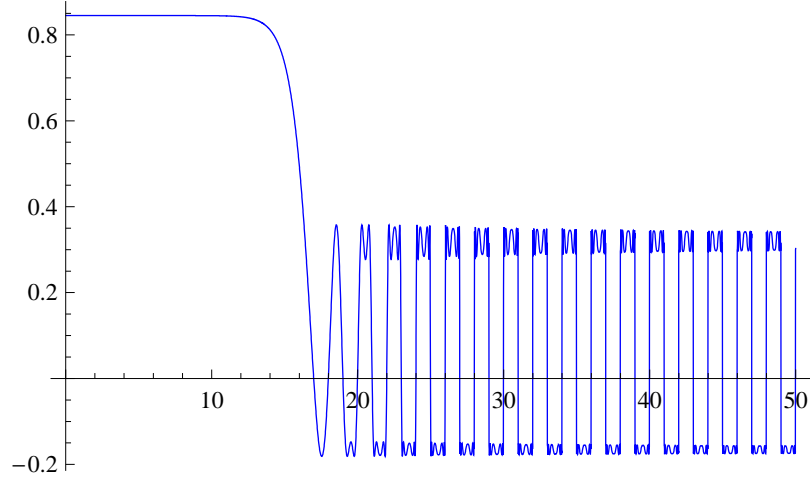


Figure 6: Rolling tachyon field at the second bifurcation in p -adic string with $p = 2$ and $r = 1 - \sqrt{6}$. The initial condition is set to $\phi = 1 - 10^{-9}$ at $X^+ = 0$.

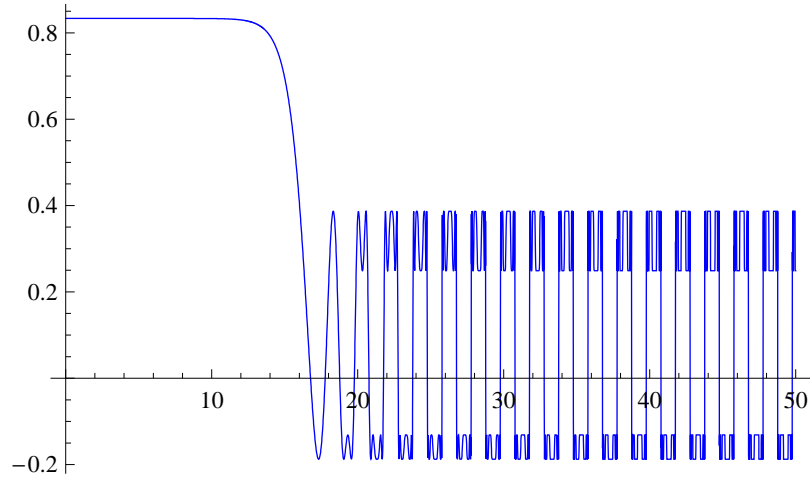


Figure 7: Rolling tachyon field for p -adic string with $p = 2$ and $r = -3/2$. The initial condition we have set $\phi = 1 - 10^{-9}$ at $X^+ = 0$. The discrete system has length-four cyclic attractor.

7 Comments on time-like tachyon rolling in string field theory

It has been known for quite some time that ordinary time-like rolling in OSFT triggered by $e^{X^0/\sqrt{\alpha'}}$ is plagued by oscillations with an amplitude that grows exponentially with a square in the exponent [26]. There is an easy way to understand this behavior. Consider for simplicity again p -adic model with an equation of motion $p^{-\alpha'}\square\phi = \phi^p$ and set $p = 2$ and $\alpha' = 1$. The solution can be constructed in the form of power series in e^t

$$\phi(t) = 1 + \sum_{n=1}^{\infty} a_n e^{nt}. \quad (7.1)$$

The coefficients a_n are given by a simple recursion

$$a_n = \frac{1}{2^{n^2} - 2} \sum_{k=1}^{n-1} a_k a_{n-k}. \quad (7.2)$$

For the initial condition we take $a_1 = -1$, so that the solution starts rolling towards the true vacuum. Any other negative value can be absorbed by a shift of t . The dominant contribution to the sum comes from the terms $k \approx n/2$, and therefore the leading asymptotic behavior of the coefficients is $a_n \sim 2^{-2n^2}$. The coefficients decay very rapidly making the series (7.1) converge for all times. Let us try to understand qualitatively the origin of the wild oscillations that the tachyon field undergoes once it is past the true vacuum. This is best illustrated by a function

$$f(t) = \sum_{n=0}^{\infty} (-1)^n e^{-\frac{1}{2}n^2} e^{nt} \quad (7.3)$$

which captures the leading behavior of the coefficients. One can easily derive a replication formula

$$f(t+1) = 1 - e^{t+\frac{1}{2}} f(t). \quad (7.4)$$

From that it follows that after period of time equal 1, the function (once sufficiently large) changes sign and its amplitude gets multiplied by a factor $e^{t+\frac{1}{2}}$. Hence the amplitude of the ever-growing oscillations goes as $e^{\frac{1}{2}t^2}$, while the period equal to 2 stays constant.

We remind that the reader at this point, that for the logistic p -adic model we found the coefficients decaying less rapidly $b_n \sim n^{-n \log_2(2-r)}$, although still leading to a series with infinite radius of convergence. In the limit $r \rightarrow 0$ the coefficients go roughly like $1/n!$ and the series $\sum (-1)^n \frac{1}{n!} e^{nx} = e^{-e^x}$ has radically different asymptotics. Empirically, the rate of decay of the coefficients is linked to the asymptotic behavior of the series, although we have not tried to make this statement precise.

In a recent paper Ellwood made an interesting observation. Assuming (perhaps based on mini-superspace intuition) that for large x^0 one can replace $e^{X^0/\sqrt{\alpha'}}$ operators by the exponential

of the zero mode $e^{x^0/\sqrt{\alpha'}}$, he observed that the rolling tachyon solution constructed recently in [15, 16] tends to the tachyon vacuum constructed by the second author in [14]. Our attempts to define the limit that Ellwood is assuming to exist failed in the time-like case, nevertheless we showed in Sec. 4 that such a limit exists in the light-like case.

Let us illustrate what happens for the time-like rolling generated by e^{X^0} in the \mathcal{L}_0 basis. The X^0 -dependent coefficient of the tachyon $c_1|0\rangle$ is given by an integral

$$\sum_{n=1}^{\infty} \lambda^n e^{nX^0/\sqrt{\alpha'}} \left(-\frac{\pi}{2}\right)^{n-1} \int_0^1 \int_0^1 \dots \int_0^1 \prod_{i=1}^{n-1} dr_i \left(\sum_{k=1}^{n-1} r_k\right)^L \prod_{1 \leq i < j \leq n-1} (x_i - x_j)^2 \quad (7.5)$$

The $n-1$ -dimensional integral can be shown to go as a square of the so called superfactorial $1!2! \dots (n-1)!$. Such a highly divergent series can't be summed by any means known to us. One could try to use Padé approximation, it gives a finite answer, but not the one we would expect. The problem can be roughly modeled on the tachyon profile given by an asymptotic expansion

$$\phi(t) = \sum_{n=1}^{\infty} (-)^{n-1} e^{\frac{1}{2}n^2} e^{nt}. \quad (7.6)$$

The above toy-series was chosen so that we can formally derive a simple functional equation

$$\phi(t+1) = 1 - e^{-\frac{1}{2}-t} \phi(t). \quad (7.7)$$

This is consistent with large time asymptotics given by

$$\phi(t) = \sum_{n=0}^{\infty} (-)^n e^{\frac{1}{2}n^2} e^{-nt} \quad (7.8)$$

which can be derived either by imposing the solution (minus one half) to be an odd function of time, or by replacing $e^{\frac{1}{2}n^2}$ with $e^{\alpha n^2}$ and perturbing formally around $\alpha = 0$. In either case the solution at late times converges to 1. The same conclusion is reached by solving numerically the functional equation (7.7). On the other hand, one can attempt to resum the divergent series using the Padé approximation with polynomials in the numerator and denominator of the same order. Quite remarkably this approximation is tractable analytically and one obtains for the asymptotic value $1 - \theta_4(\frac{i}{4}, e^{-3/2}) = 0.49557\dots$. At this point we have to conclude that the Padé approximation can sometimes fail to produce a meaningful answer, especially when the series is too divergent.

What can be done about the highly divergent series (7.5) to prove or disprove Ellwood's assertion that the rolling tachyon solution approaches the tachyon vacuum? Due to the complicated form of the coefficients, it is hard to find a relevant functional equation. Padé approximation could be used, but seems to produce some spurious answer. One trick that leads to the purported answer is to use perturbation theory around the light-like rolling. Technically, one could

rewrite

$$\prod_{1 \leq i < j \leq n-1} (x_i - x_j)^2 = e^{2 \sum_{1 \leq i < j \leq n-1} \log(x_i - x_j)}, \quad (7.9)$$

replace 2 in the exponent by a parameter α and perturb around $\alpha = 0$. Then at each order in α the solution at late times goes to zero, the true vacuum. However, we would like to caution, that this is just one way of dealing with extremely divergent series that happens to give us the answer we want. Padé approximation gives us another answer.

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A Some Bernoulli number identities

The Bernoulli numbers are defined via

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}. \quad (\text{A.1})$$

At certain intermediate steps during our computation we have found a useful identity

$$\begin{aligned} \left(\frac{z}{e^z - 1} \right)^M &= \sum_{n=0}^{\infty} \sum_{\{k_i | \sum k_i = n\}} \binom{n}{k_1, \dots, k_M} B_{k_1} \dots B_{k_M} \frac{z^n}{n!} \\ &= \frac{(-1)^{M-1}}{(M-1)!} \sum_{n=0}^{\infty} \sum_{k=1}^M |S_M^{(k)}| B_{n-M+k} \prod_{\substack{j=0 \\ j \neq M-k}}^{M-1} (n-j) \frac{z^n}{n!}. \end{aligned} \quad (\text{A.2})$$

In the second line $|S_M^{(k)}|$ are the unsigned Stirling numbers of the first kind defined as the number of permutations of M elements containing exactly k permutation cycles. This identity was first discovered by Lucas in 1878 [82], see also [83] for a modern perspective.

Simplest nontrivial identity of this sort is the celebrated Euler identity

$$\begin{aligned} \left(\frac{z}{e^z - 1} \right)^2 &= \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} B_k B_{n-k} \frac{z^n}{n!} \\ &= - \sum_{n=0}^{\infty} ((n-1)B_n + nB_{n-1}) \frac{z^n}{n!}. \end{aligned} \quad (\text{A.3})$$

The above identity can be used to derive for instance

$$\left(\frac{d}{d\alpha} \right)^n \frac{(e^\alpha - 1)^2 - \alpha^2 e^\alpha}{(e^\alpha - 1)^4} \Big|_{\alpha=0} = \frac{n-3}{6} B_{n+2} + \frac{n-2}{2} B_{n+1} + \frac{n-1}{3} B_n \quad (\text{A.4})$$

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